Neighborhood Homogeneous Labeling of Graphs

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Outline

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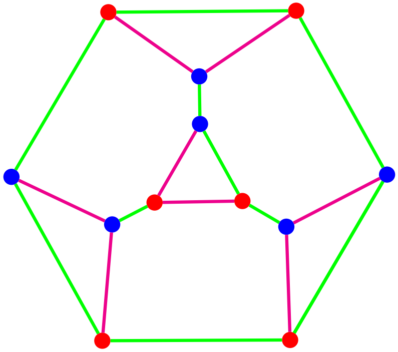
Outline

Illustration of the Problem



Can we color (label) a graph (edges/vertices) so that every neighborhood “looks the same”?

Definitions: Labeled Graphs

- Graphs will be undirected and simple. $G = (V, E, \varphi_V, \varphi_E)$. Later we will allow multiple edges between vertices (multi-graphs).
- $\varphi_\bullet$ are labeling functions.
- We have k_v vertex labels $\mathcal{L}_V$ and k_E edge labels $\mathcal{L}_E$.
  - $\varphi_V : V \to \mathcal{L}_V$.
  - $\varphi_E : E \to \mathcal{L}_E$.
- The closed neighborhood of a vertex v is $N[v] = \{w \in V | vw \in E\} \cup \{v\}$.
- We will denote by $\Omega(N[v])$ the induced subgraph of $N[v]$ in G, and by $E[N[v]]$ the edge set of $\Omega(N[v])$.

Definitions: Homogeneous Labeling

- A graph $G = (V, E, \varphi_V, \varphi_E)$ is $(a, b; c, d)$-NL (neighborhood labeled) if for every vertex u:
  1. $k_V = a$.
  2. $|\{v \in N[u] | \varphi_V(v) = x\}| = b$ for all $x \in \mathcal{L}_V$.
  3. $k_E = c$.
  4. $|\{uv \in E[N[u]] | \varphi_E(uv) = y\}| = d$ for all $y \in \mathcal{L}_E$.
- Thus:
  - there are a vertex labels and each appears b times in every closed neighborhood.
  - there are c edge labels and each appears d times in every closed neighborhood.
- We use NVL and NEL for graphs that are vertex/edge neighborhood homogeneous labelable.

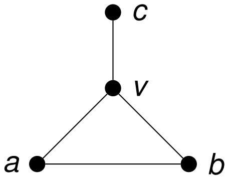
Outline

Some Easy Facts

- Let G be $(2, k; 2, s)$-NL.
  1. $|N[v]| = 2k$ and $|E[N[v]]| = 2s$.
  2. G is $(2k - 1)$-regular.
  3. G contains a 3-cycle and so G is not bipartite.
  4. If G is a complete graph, then $G \cong K_{4m}$ for some $m \geq 1$.
  5. If G is connected, then $\Omega(N[v])$ is not a clique unless G is complete.
- It is easy to generalize these to the general case.
- We will consider $k = s = 2$ from now on.

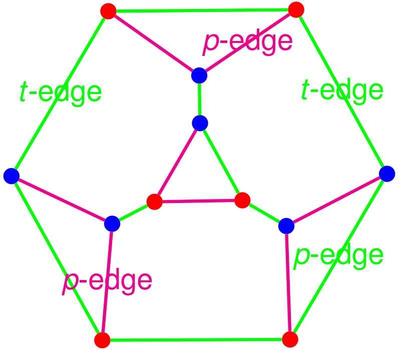
The $(2, 2; 2, 2)$-NL Case

1. G is 3-regular with $|V|$ a multiple of 3.
2. For each $v \in V$, $N[v]$ is isomorphic to:



3. The vertex c is on a three cycle in G_c. Thus every edge is either on a 3-cycle, or connecting two 3-cycles. Call the former a p-edge and the latter a t-edge.
4. If two p-edges are on an ℓ-cycle with $\ell > 3$, they do not have a common end vertex. The same is true for t-edges. Hence, p-edges and t-edges alternate on such a cycle.

Illustration of the Results



More Results

- The order of a $(2,2;2,2)$-NL graph is a multiple of 6.
- The length of a cycle is either 3 or even.
- If a graph G is $(2,2)$-NVL and every neighborhood has 4 edges, then G is $(2,2)$-NEL and hence $(2,2;2,2)$-NL.
- Every $(2,2)$-NVL graph that is not K_4 is $(2,2;2,2)$-NL.

Outline

Collapsing Characterization

- Given a $(2, 2; 2, 2)$-NL graph, denote by $\widetilde{G}$ the multi-graph where every triangle is collapsed to a node.
- $\widetilde{G}$ is an $(2, 2)$-NLV multi-graph.
- If in a loopless $(2, 2)$-NVL multi-graph $\widetilde{G}$ each vertex is replaced by a 3-cycle, connecting each incident edge to a unique vertex, the resulting graph is $(2, 2; 2, 2)$-NL.
- This “expanding by a 3-cycle” is a clique-expansion of the graph.

Reduction Theorem

If G is a $(2, 2)$-NLV multi-graph with $n > 4$ vertices, then there is a (possibly disconnected) multi-graph G^* which is $(2, 2)$-NVL with $n - 4$ vertices, constructed by removing 4 vertices from G (and adding edges).

Outline

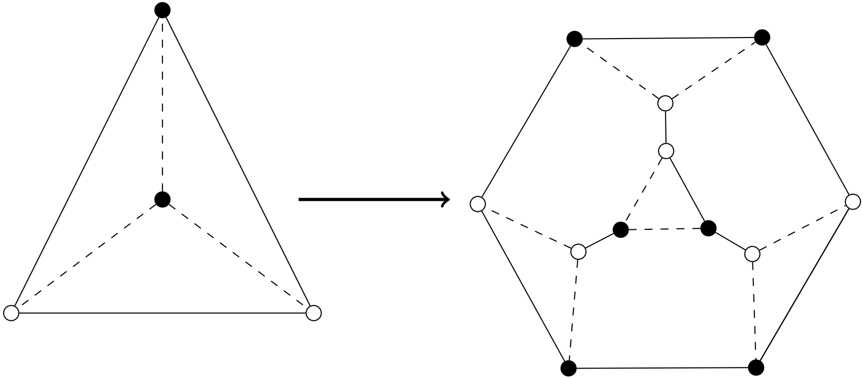
Definitions and Simple Facts

- Let A be the adjacency matrix of G:

$$a_{ij} = 1 \iff ij \in E.$$

- The spectrum of the graph G is the set of eigenvalues of G.
- We represent the spectrum in increasing order.
- Since G is undirected, A is symmetric and real, so the eigenvalues are real.
- If G is r-regular and $|V| = n$, then $\lambda_n = r$.
- Consider a clique expansion of a 3-regular graph.

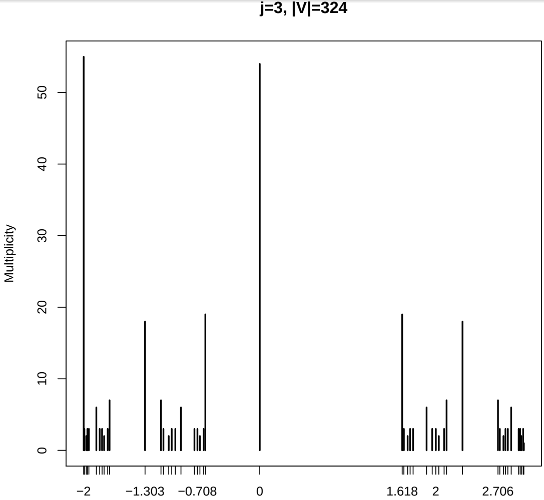
Clique Expansion



Spectra

- Start with $G_{-1} = K_4$ and let G_j be the clique-expansion of G_{j-1}. Then:
  - $n_j = |V| = 4 * 3^{j+1}$ (obvious).
  - The maximum eigenvalue is 3 (the graph is 3-regular) and the minimum is -2.
  - The 0 eigenvalue has multiplicity $2 * 3^j$, -2 has multiplicity $2 * 3^j + 1$, and 3 has multiplicity 1.
  - If λ is an eigenvalue of G_j then it is an eigenvalue of G_{j+1}.
  - Conjecture (from observations): Let $m^{j_1}, ..., m^{j_k}$ be the multiplicity of the distinct values of the eigenvalues of G_j. Then the m^{j_i} are symmetric about the value for the 0 eigenvalue, except for the minimum and maximum values.

Example



Outline

Discussion

- This work came out of work on detecting anomalies in graphs – small regions in which the vertices have different connectivity than the rest of the graph. If the graphs indicate communications, there may be attributes on the edges (topic of discussion) and/or attributes on the vertices (information about the individual). We are interested in people that communicate in a different pattern on different topics than the rest of the graph.
- Thus the question: how homogeneous can a graph be?
- We have developed the theory of $(2, 2; 2, 2)$-NL graphs, and given a couple of characterization theorems.
- These graphs have interesting spectral properties, which come from the fact that they are clique-expansions.

Future Work

- Extend to general $(a, b; c, d)$-NL graphs.
  - What are the conditions such that a clique-expand theorem is possible?
  - For given a, b, c, d, how many distinct neighborhoods are possible for $(a, b; c, d)$-NL graphs?
- What happens if we change from equality to proportion:
  - Say a graph is (a, b)-PNVL (prortionally NVL) if for every vertex u and label x $|\{v \in N[u] | \varphi_V(v) = x\}||N[u]| = b$.
- What are the properties of PNVL graphs? Is it true (as seems likely) that a PNVL graph can have neighborhoods of different sizes?