Detecting Activity Changes in Graphs

David J. Marchette

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Interface 2008

Outline

Motivation

Definitions and Model

Change Detection

Conclusions

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- Change Detection
- Conclusions

Social Networks

This builds on work presented at QMDNS on Wednesday.
Many of the slides will be repeats from that talk.

- A model of the relationships between entities.
- Also used to study insurgent groups, terrorist cells, etc.
- Relates actors (nodes in the network) through relationships (edges in the network).
- Typically used for small groups, with full knowledge of all links.

Covert Networks

- Actors have a vested interest in not being observed.
- Networks may be very large.
- The networks change in time.
- An actor may try to hide (change email address, change phone number, start calling themselves Colonel Guapa).
- In this work we are interested in cases where the actor changes their activity profile. Can we detect that they have made a significant change?

Methodology

- Assume the existence of a "social space" $\mathcal{S}$ which controls the structure of the network.
- The probability of an edge in the network is a function of the "closeness" of the nodes in $\mathcal{S}$.
- The social space provides a framework from which inference can be performed.

Social Space

- Early work reported by Hoff et al in JASA.
- Model based on location:
  - Probability of an edge between v_i and v_j a function of their distance in social space.
  - Several variations proposed.
- Versions of the Exponential Random Graph Models (ERGMs) (Hunter et al, JASA 2008) can be thought of in terms of a “social space”.
- We will discuss a “social space” model that has a simple least squares algorithm for fitting the parameters, which can be used on large graphs (thousands to tens of thousands of nodes or more).

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Graph Definitions

- A graph is a pair (V, E) where V is a set (vertices) and E is a collection of unordered pairs of vertices (edges).
- We can consider directed graphs (V, A) where A (arcs or arrows) are ordered pairs.
- The order of the graph is $|V|$ and the size of the graph is $|E|$ (or $|A|$ in the case of directed graphs (digraphs)).
- Vertices are sometimes called “nodes” or “actors”.
- Edges are sometimes called “links” or “relations”.
- The adjacency matrix $A = (a_{ij})$ is the $|V| \times |V|$ binary matrix with a 1 in those places where an edge occurs in the graph.

Probabilistic Framework

- We place a probability structure on the network.
- This means we fit a **generative** model to the graph.
- This allows us to estimate the probability of a missing (unknown) link.
- We can bring node attributes into the model.
- We are essentially choosing the “most likely” graph given the model assumption and the observed edges.

Random Dot Product Graphs

- Each vertex v_i has associated with it a vector x_i.
- Place an edge $v_i v_j$ between vertices v_i and v_j with probability proportional to $x_i x_j$, the dot product of x_i and x_j.
- Thus $p_{ij} = f(x_i x_j)$. We'll use the threshold function for f:

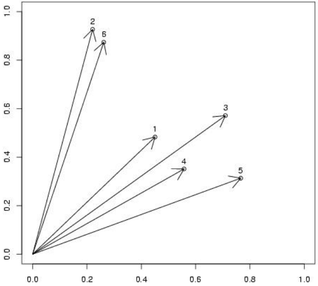
$$f(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

- The edges in the random graph are no longer independent.
- We need to estimate the x_i from the observed graph.
- We can extend the model to directed graphs by having in- and out-vectors x_i^I and x_i^O with p_{ij} proportional to $x_i^O x_j^I$.

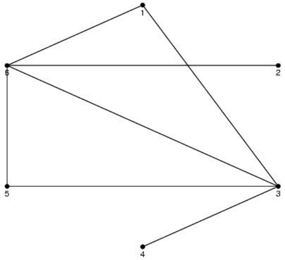
$\mathcal{S}$

- Each vertex v_i has associated with it a vector $x_i \in \mathcal{S}$.
- The proximity (as measured by the dot product) of two vectors controls the probability of an edge.
- Thus $\mathcal{S}$ is the space which defines the random graph that we observe.

$\mathcal{S}$



$\mathcal{G}$



Linear Algebra (Least Squares)

Note that if we want to find the vectors U which best "match" the adjacency matrix A (best in Frobenius norm), then the singular value decomposition: $A = UDV'$ almost works (the problem is the diagonal).

1. Set $D = \text{diag}(0)$.
   1.1 $s = \text{svd}(A + D)$.
   1.2 $X = U$, scaled by the singular values.
   1.3 $D = \text{diag}(XX')$.
2. Repeat 1–3 until convergence.
3. Return X.

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Aliases

- Given two graphs G_t and G_{t+1}.
- Suppose we know some of the vertices are shared by these graphs (and which ones they are).
- There is one vertex in G_{t+1} that we have not seen before.
- Assuming that this vertex appeared in G_t with a different label, can we determine this vertex?
- For change detection, we want to compare each vertex at time t with itself at time $t-1$, and all other vertices at this time.
- If the vertex is less like itself than it is like the others, it has changed.

Aliases

- Setup:
  - Two graphs, $G_t = (V \cup U_t, E_t)$ and $G_{t+1} = (V \cup U_{t+1}, E_{t+1})$.
  - All vertices are labeled (email addresses).
  - Vertices in V are named (individual associated with the address).
  - Vertices in U_i are not named.
- Want to associate the names to the vertices in U_{t+1}.

Methodology

- Assign the name to vertex u whose vector x_v is closest to the vector x_u.
- Optimize:

$$(X, Y_1, Y_2) = \arg\min_{X, Y_1, Y_2} \left\| \left(\begin{pmatrix} X \\ Y_1 \end{pmatrix} \begin{pmatrix} X \\ Y_1 \end{pmatrix}^T \right)_0 - A_1 \right\|_F + \left\| \left(\begin{pmatrix} X \\ Y_2 \end{pmatrix} \begin{pmatrix} X \\ Y_2 \end{pmatrix}^T \right)_0 - A_2 \right\|_F ,$$

- M_O means M with the diagonal replaced with zeros.
- Thus, we are attempting to fit a set of vectors to the known and a set each for the unknown in the two graphs. Fitting to the knowns constrains the Y_i to lie in the same space.

The Setup

- Input A_1, A_2, the adjacency matrices of the graphs corresponding to the vertices (V, U_i).
- Set B to be the average of $A_1[V]$ and $A_2[V]$, the blocks corresponding to V.
- Set $N = n + n_1 + n_2$.
- Set A to be the $N \times N$ matrix with first $n \times n$ block equal to B, and blocks $A[V, U_i] = A_i$, $A[U_i, V] = A_i'$.

$$A = \begin{pmatrix} \frac{A_1[V,V]+A_2[V,V]}{2} & A_1[V, U_1] & A_2[V, U_2] \\ A_1[U_1, V] & A_1[U_1, U_1] & Y' \\ A_2[U_2, V] & Y & A_2[U_2, U_2] \end{pmatrix}$$

where Y is the dot product of vectors derived from U_1 and U_2.

Fitting the Alias

1. Setup as described previously.
2. Set $D = 0_{N \times N}$.
3. Set the first $n \times n$ block of D equal to the the dot product of the result of running the least squares Algorithm on B.
    3.1 While(Not Converged)
    3.2 $Y = g_d(A + D)$ (scaled U from the svd).
    3.3 Set the unknown entries of D (such as those corresponding to $U_1 \times U_2$) to the dot products of the appropriate parts of Y.
4. Output$Y$

- Use the vectors to find the alias: closest named vector to the one associated with the alias.

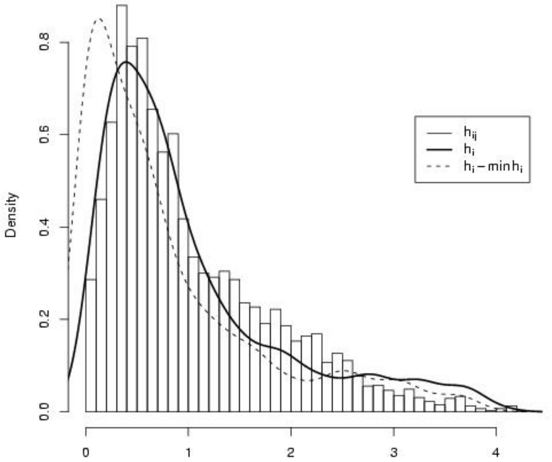
Change Detection

- Run the alias detection code with $U_t = U_{t-1} = \{v_i\}$ and with $U_t = \{v_i\}$, $U_{t-1} = \{v_j\}, j \neq i$.
- Use the distance to the correct vertex compared to those with other vectors to determine whether a change has occurred.
- We will look at some simulations illustrating this:
  1. Consider Erdös-Renyí graphs: the distribution of the distances for the correct vertex should match those for the others.
  2. Consider RDPG graphs: the distribution of the distances for the correct vertex should be stochastically smaller than those for the others.
- The simulations will bear this out.
- Note that one need not compare all $\binom{n}{2}$ pairs in order to obtain an estimate of the alternative: a subsample is sufficient for larger graphs, or one could compute this once off-line if one knew the basic model for the graphs.

Change Detection Simulation: ER

Erdös-Renyí graphs (no change): $|V| = 10, p = 0.5$.

1 graph before the purported change point, 1 graph after.



Change Detection Simulation: ER

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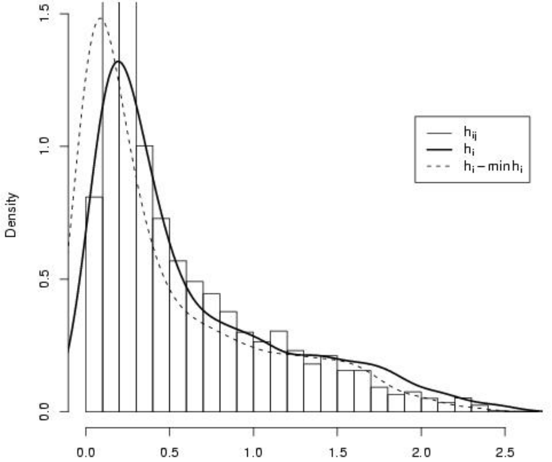
10 graphs before the purported change point, 10 graphs after.



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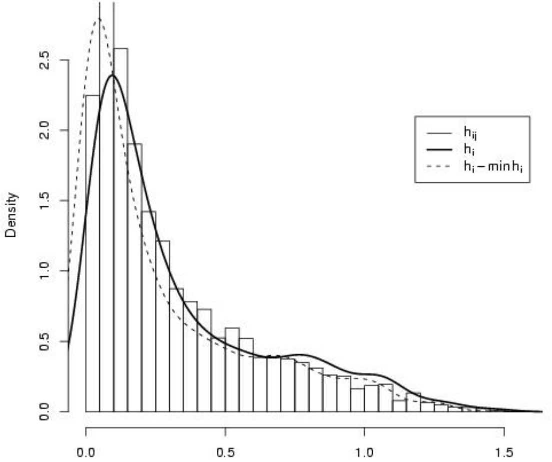

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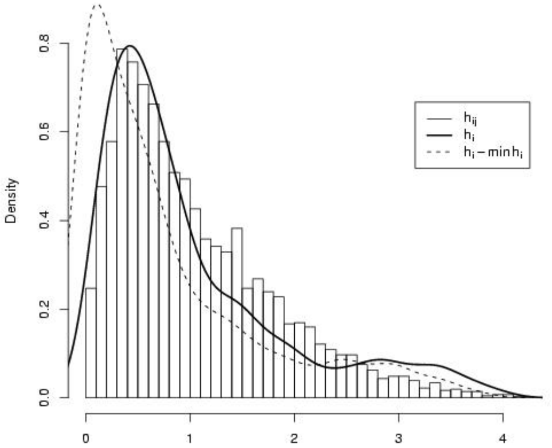

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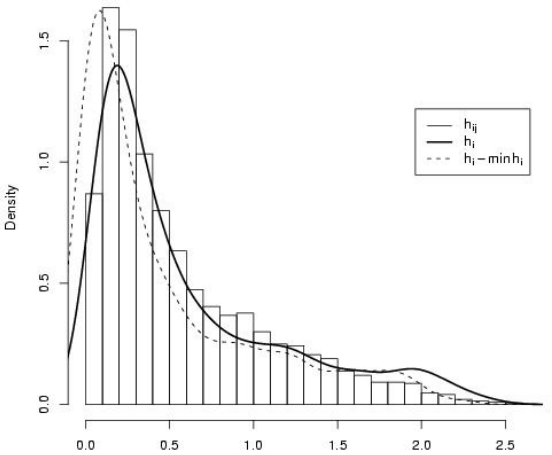

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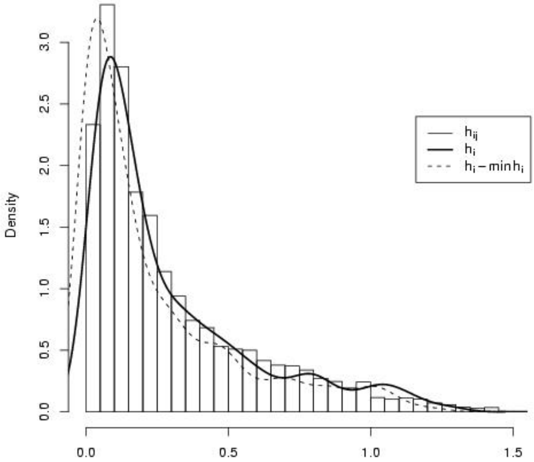

Change Detection Simulation: RDPG

Random Dot Product graphs (no change): $|V| = 10$.
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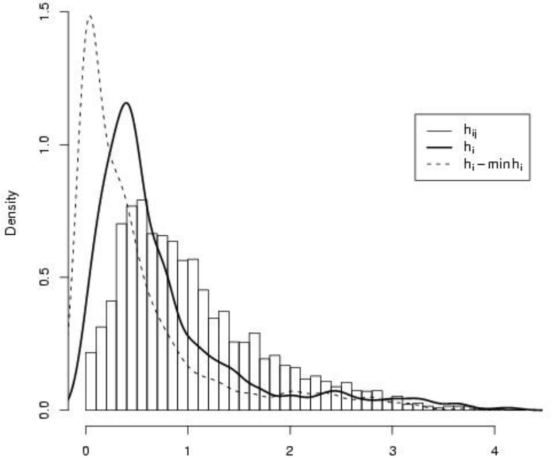

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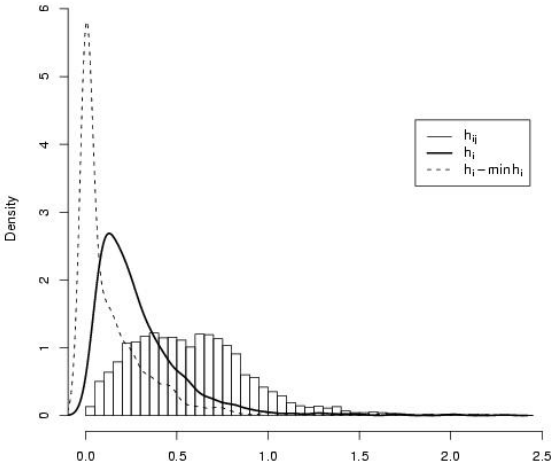

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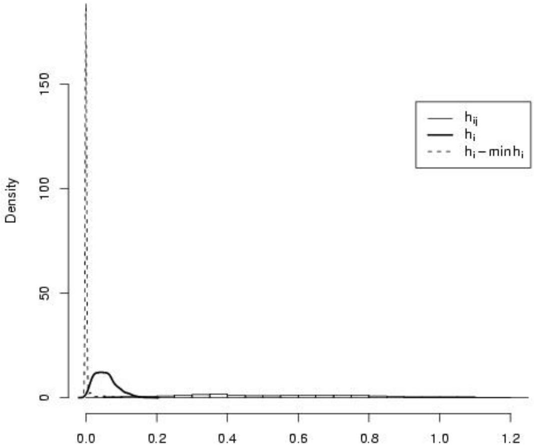

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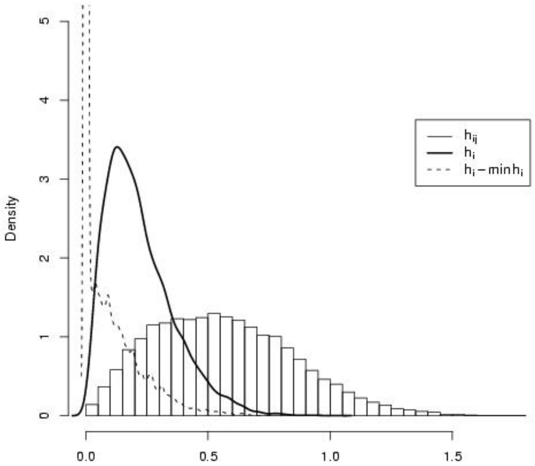

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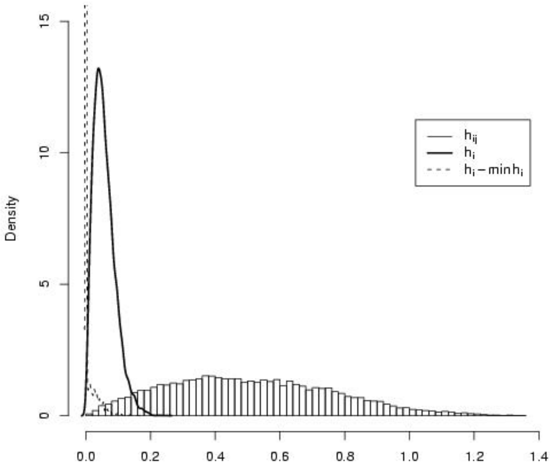

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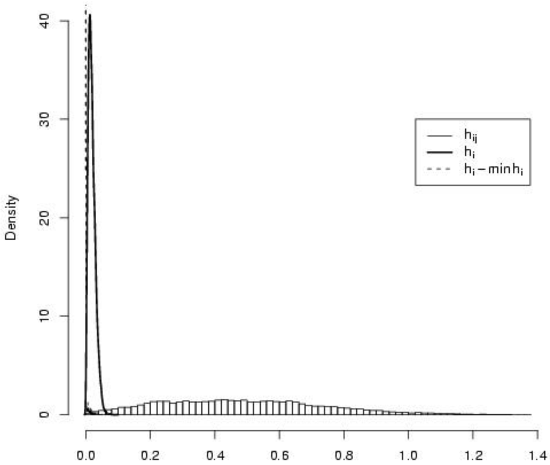
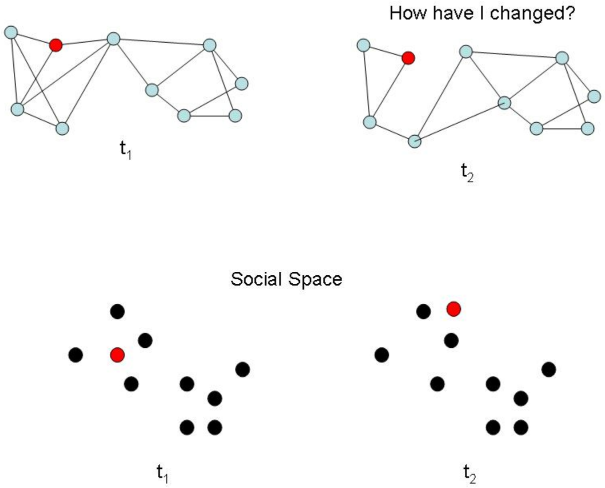


Cartoon



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- Social space provides a mechanism for modeling and inference on graphs and time series of graphs.
- Dot product graph model is simple, but easy to fit using linear algebra.
- Sparse matrix approaches can make this efficient.
- It is possible to add covariates (measurements at the nodes) into the model and still use the linear algebra approach, but this work is preliminary.
- Changes in the graphs can be detected at the vertex level, and we have a natural estimate of the distribution under the alternative (comparison to the v_i vs v_j distances).

Questions?

Contact Information: dmarchette@gmail.com