

# Splitting cycles in graphs<sup>☆,☆☆</sup>

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## Abstract

The goal of this paper is to describe a sufficient condition on cycles in graphs for which the edge ideal is splittable. We give an explicit splitting function for such ideals.

*Keywords:* Edge Ideal, Monomial Ideal, Betti Number, Splittable Monomial Ideal

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## 1. Algebraic preliminaries

We begin by recalling some definitions and theorems from commutative algebra. What follows in this introductory section is contained in the paper of Eliahou and Kervaire ([1]). Let  $I$  be a monomial ideal in a polynomial ring whose coefficient field has characteristic zero. We will denote by  $\mathcal{G}(I)$  a minimal set of generators for  $I$  (in fact, *the* minimal set of generators). That is,  $\mathcal{G}(I)$  is the set of all monomials in  $I$  that are not proper multiples of any monomial in  $I$ . We note that if  $I$  is the edge ideal of a simple graph,  $G$ , then the monomials corresponding to the edges of  $G$  are such a minimal set of generators.

**Definition 1.** *A monomial ideal,  $I$ , is splittable if  $I$  is the sum of two nonzero monomial ideals,  $J$  and  $K$ , that is,  $I = J + K$ , such that*

- (1)  $\mathcal{G}(I)$  is the disjoint union of  $\mathcal{G}(J)$  and  $\mathcal{G}(K)$
- (2) there is a splitting function

$$\begin{aligned}\mathcal{G}(J \cap K) &\rightarrow \mathcal{G}(J) \times \mathcal{G}(K) \\ w &\mapsto (\phi(w), \psi(w))\end{aligned}$$

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satisfying

- (a) for all  $w \in \mathcal{G}(J \cap K)$ ,  $w = \text{lcm}(\phi(w), \psi(w))$ ,
- (b) for every subset  $S \subset \mathcal{G}(J \cap K)$ , both  $\text{lcm}(\phi(S))$  and  $\text{lcm}(\psi(S))$  strictly divide  $\text{lcm}(S)$ .

If  $J$  and  $K$  satisfy both the above properties, then we say that  $I = J + K$  is a splitting of  $I$ .

The following theorem, which relates the minimal free resolution of each ideal in the splitting to the minimal free resolution of their sum (see [1]), is of central importance in what follows.

**Theorem 2 (Eliahou-Kervaire).** *Suppose  $I$  is a splittable monomial ideal with splitting  $I = J + K$ . Then for all  $i, j \geq 0$ ,*

$$\beta_i(I) = \beta_i(J) + \beta_i(K) + \beta_{i-1}(J \cap K).$$

## 2. Splitting cycles

We now want to apply the notion of splitting monomial ideals to the specific case of edge ideals associated to simple (and undirected) graphs. In particular, we want to “split along” a particular subgraph. To this end, we recall some standard definitions from graph theory and we establish the notation and conventions that we will use in what follows.

If  $G$  is a simple graph with edge set  $E$  and vertex set  $V = \{x_1, \dots, x_n\}$ , then the *edge ideal* of  $G$  is the (square-free quadratic) monomial ideal  $\mathcal{I}(G) = \{x_i x_j \mid \{x_i, x_j\} \in E\}$  contained in the polynomial ring  $R = k[x_1, \dots, x_n]$ . We will hereafter assume that the field  $k$  has characteristic zero (so the reader may feel free to take  $k$  to be any of  $\mathbb{Q}$ ,  $\mathbb{R}$ , or  $\mathbb{C}$ ).

We now suppose that  $G$  is a graph of order  $n$  that contains an (induced)  $k$ -cycle with no chords, where  $k \geq 4$ . We will adopt the following notation. Let  $G = (V, E)$ , where  $V$  are the vertices and  $E$  are the edges in the graph  $G$ . Let  $U = \{u_1, \dots, u_k\}$  denote the vertices in the  $k$ -cycle and  $W = \{w_1, \dots, w_{n-k}\}$  denote the vertices of  $G$  not contained in the cycle. Then,  $V = U \sqcup W$ . Now, let  $E_U = \{u_1 u_2, u_2 u_3, \dots, u_{k-1} u_k, u_k u_1\}$ ; i.e.,  $E_U$  is the  $k$ -cycle contained in  $G$ . Also denote  $E_W = \{w_p w_q \mid w_p, w_q \in W \text{ and } w_p w_q \in E\}$ . Finally, let  $E_X = \{u_i w_p \mid u_i \in U \text{ and } w_p \in W \text{ and } u_i w_p \in E\}$ . Just as we wrote for the vertices of  $G$ , we can write the edges as a disjoint union,  $E = E_U \sqcup E_W \sqcup E_X$ . We will often denote the  $k$ -cycle by  $\mathcal{C}_k = (U, E_U)$ . Recall, also, that if  $H \subset G$  is a subgraph, then by the *complement* of  $H$  in  $G$  we shall mean the subgraph of  $G$  consisting of all vertices of  $G$  and those edges in  $G$  that are not in  $H$ . Finally, to simplify the notation, we will adopt the convention in what follows that *any subscript appearing on a “ $u$ ” labeled vertex (i.e., a vertex in  $U$ ) will be understood to be (mod  $k$ )*.

We wish to parallel the work that Hà and Van Tuyl did on splitting edges in a graph ([2]). In what follows, we prove a sufficient condition for a cycle to be splitting. We begin by making the obvious definition.

**Definition 3.** We denote by  $G \setminus \mathcal{C}_k$  the complement of the  $k$ -cycle in  $G$  (i.e.,  $G \setminus \mathcal{C}_k = (V, E_W \cup E_X)$ ). Then, a  $k$ -cycle,  $\mathcal{C}_k$ , is a splitting cycle if  $\mathcal{I}(G) = \mathcal{I}(\mathcal{C}_k) + \mathcal{I}(G \setminus \mathcal{C}_k)$  is a splitting.

Now, let  $G$  be a graph containing an induced  $k$ -cycle with no chords. We denote the edge ideal of the graph,  $G$ , by  $I := \mathcal{I}(G)$ , and we then let  $J := \mathcal{I}(\mathcal{C}_k)$  and  $K := \mathcal{I}(G \setminus \mathcal{C}_k)$ .

**Lemma 4.** With the notation above,  $I = J + K$ .

PROOF. The proof of this lemma is essentially a tautology. Note that  $\mathcal{G}(J) = E_U$  and that  $\mathcal{G}(K) = E_W \sqcup E_X$ . Thus,  $\mathcal{G}(I) = \mathcal{G}(J) \sqcup \mathcal{G}(K)$ , and the result is immediate.  $\square$

We now want to characterize the minimal generating set of the intersection of the ideals  $J$  and  $K$ ,  $\mathcal{G}(J \cap K)$ . Before stating the lemma we recall some facts about the minimal generating sets of monomial ideals. First, if  $J$  and  $K$  are monomial ideals, then the intersection,  $J \cap K$  is generated by the set

$$\{\text{lcm}(f, g) \mid f \in J, g \in K\}$$

(see, for example, [3]). Then, one reduces this generating set to a minimal one by checking for pairwise divisibility.

**Lemma 5.** Let  $I, J$ , and  $K$  be as defined above. Furthermore, let

$$\begin{aligned} A &= \{u_i u_{i+1} w_p \mid u_i u_{i+1} \in E_U \text{ and } (u_i w_p \in E_X \text{ or } u_{i+1} w_p \in E_X)\} \\ B &= \{u_i u_{i+1} u_j w_p \mid u_i u_{i+1} \in E_U \text{ and } u_j w_p \in E_X \text{ and } u_i u_{i+1} w_p, u_{i+1} u_j w_p \notin A\} \\ C &= \{u_i u_{i+1} w_p w_q \mid u_i u_{i+1} \in E_U \text{ and } w_p w_q \in E_W \text{ and } u_i u_{i+1} w_p, u_i u_{i+1} w_q \notin A\} \end{aligned}$$

Then  $\mathcal{G}(J \cap K) = A \sqcup B \sqcup C$ .

PROOF. We provide a constructive proof of this lemma. To construct the set  $\mathcal{G}(J \cap K)$  one begins by considering all possible products of elements of  $\mathcal{G}(J)$  and  $\mathcal{G}(K)$ . From this set, remove any duplicate monomials. Then, using the characterization of the minimal generating set as the maximal set such that no element divides any other, we reduce this collection.

We begin by identifying the monomials in our proposed  $\mathcal{G}(J \cap K)$  that contain a square (i.e., a  $u_i^2$  for some  $i$ ). We replace these degree four monomials with the degree three monomial in which  $u_i$  replaces the  $u_i^2$ . Note that there can be no terms containing a  $w_p^2$  since  $\mathcal{G}(J)$  contains only  $u$  variables and our graph is assumed to be simple (so there is no  $p$  such that  $w_p^2 \in \mathcal{G}(K)$ ). Then the degree three monomials are exactly the set  $A$ .

Continuing, we next eliminate the redundant degree four monomials that are of degree three in the  $u$  variables and degree one in the  $w$  variables. These are precisely the the degree four monomials that are contained in the ideal generated by  $A$  (i.e., that are multiples of some element of  $A$ ). So, we want to remove

redundant monomials of the form  $u_i u_{i+1} u_j w_p$ ; that is, we want to determine when  $u_i u_{i+1} u_j w_p$  is a multiple of some  $u_k u_{k+1} w_q \in A$ , some  $k$ . First, note that we must have  $p = q$ . Suppose not; i.e., suppose that  $w_p \neq w_q$ . We are assuming that  $u_k u_{k+1} w_q \mid u_i u_{i+1} u_j w_p$ . Now  $U \cap W = \emptyset$  implies  $u_k \nmid w_p$  (and similarly  $u_{k+1} \nmid w_p$ , of course) and, by assumption,  $w_q \nmid w_p$ . Thus,  $u_k u_{k+1} w_q \mid u_i u_{i+1} u_j$ . However, both monomials are of degree three and  $w_q \nmid u_m$  for any  $m$ . This is a contradiction. Thus,  $p = q$ . Now,  $u_k u_{k+1} w_p \mid u_i u_{i+1} u_j w_p$  if and only if  $u_k u_{k+1} \mid u_i u_{i+1} u_j$ . Thus, either  $k = i$ , or  $k = i + 1$  and  $j = k + 1 (= i + 2)$ . This is precisely the set  $B$ .

The final step in producing  $\mathcal{G}(J \cap K)$  is to eliminate the degree four monomials that are of degree two in both  $u$  and  $w$  and that are contained in the ideal generated by  $A$  (note, we obviously needn't consider multiples of elements of the set  $B$ ). Elements of degree two in both  $u$  and  $w$  must be of the form  $u_i u_{i+1} w_p w_q$ , as they are products of elements of  $\mathcal{G}(J)$  and  $\mathcal{G}(K)$ . Thus, an element of this form is a multiple of an element of  $A$  if  $u_i u_{i+1} w_p \in A$  or  $u_i u_{i+1} w_q \in A$ . This is precisely the set  $C$ .

The very nature of our construction (i.e., eliminating redundant monomials) guarantees that the decomposition is as a disjoint union.  $\square$

We are now ready to state and prove the main proposition about splitting cycles. In the proposition immediately below, we give a sufficient condition on a cycle for the cycle to be splitting.

**Proposition 6.** *If there is no  $i \in \{1, \dots, k\}$  such that  $u_i w_p, u_{i+1} w_q \in E$  then  $\mathcal{C}_k$  is a splitting cycle. Equivalently, the cycle,  $\mathcal{C}_k$ , is splitting if it contains no adjacent vertices of degree greater than 2.*

PROOF. We first comment that the equivalence of the two statements of the proposition is immediate and obvious. So, we now proceed with the proof of the first statement in the proposition.

Using the notation introduced in Lemma 5 we first define a splitting function,  $\mathcal{G}(J \cap K) \rightarrow \mathcal{G}(J) \times \mathcal{G}(K)$ , by

$$w \mapsto (\phi(w), \psi(w)) = \begin{cases} (u_i u_{i+1}, u_i w_p) & , \quad w \in A \text{ and } u_i w_p \in E_X \\ (u_i u_{i+1}, u_{i+1} w_p) & , \quad w \in A \text{ and } u_{i+1} w_p \in E_X \\ (u_i u_{i+1}, u_j w_p) & , \quad w \in B \\ (u_i u_{i+1}, w_p w_q) & , \quad w \in C. \end{cases}$$

Now, we need to verify that our function satisfies conditions (a) and (b) given in Definition 1. Note the condition (a) is an immediate consequence of the decomposition of  $\mathcal{G}(J \cap K)$  given in Lemma 5 above. So, let  $S \subset \mathcal{G}(J \cap K)$ . Our description of  $\mathcal{G}(J \cap K)$  in Lemma 5 implies that each element of  $S$  is divisible by some  $w_p$  while no element of  $\phi(S)$  is divisible by any  $w_p$ . Thus,  $\text{lcm}(\phi(S))$  strictly divides  $\text{lcm}(S)$ .

We introduce one more notational convenience. We define the (open) neighborhood of a  $k$ -cycle,  $\mathcal{C}_k$ , to be the union of the (open) neighborhoods of the vertices in the cycle minus the vertices in the cycle itself. That is

$$\mathcal{N}(\mathcal{C}_k) := (\cup_{u_i \in \mathcal{C}_k} \mathcal{N}(u_i)) \setminus U$$

We can now conveniently characterize  $\psi(S)$  as follows:

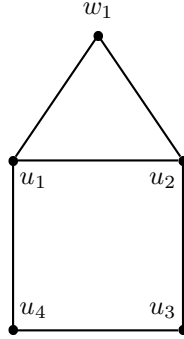
$$\psi(S) = \{u_j w_p \mid w_p \in \mathcal{N}(\mathcal{C}_k)\} \cup \{w_p w_q \mid w_p, w_q \notin \mathcal{N}(\mathcal{C}_k)\}$$

We now note that, by our assumption, if for some  $j \in \{1, \dots, k\}$  we have  $u_j w_p \in \psi(S)$ , then we must have  $u_{j-1} w_r, u_{j+1} w_s \notin \psi(S)$  for any  $r, s \in \{1, \dots, n-k\}$ . However, for any  $S \subset \mathcal{G}(J \cap K)$  we must have some  $i \in \{1, \dots, k\}$  for which  $u_i u_{i+1}$  divides  $lcm(S)$ . Thus, the divisibility is strict.  $\square$

We should emphasize that the splitting that we have constructed in the proof of Proposition 6 is valid only when the cycle has no chords. That is, if  $\mathcal{C}_k$  has an edge  $\{u_i, u_j\}$  where  $i$  and  $j$  are not consecutive integers (mod  $(k)$ ), then the decomposition of the minimal generating set of the intersection,  $\mathcal{G}(J \cap K)$ , given in Lemma 5 does not hold. Consequently the splitting function given in Proposition 6 is not actually a splitting. In fact, in this case we won't have even specified a function on  $\mathcal{G}(J \cap K)$ , as monomials of the form  $u_i u_j$  appear in  $\mathcal{G}(J \cap K)$  and our definition of the splitting function does not indicate where such terms would be mapped. Furthermore, the proof of strict divisibility given in Proposition 6 relies the presence of  $w_p$  terms for the strict divisibility.

In Proposition 6 above, we showed sufficiency of the condition on degrees of vertices to guarantee that a cycle is splitting. We now provide an example to show that the condition is not also necessary.

**Example 7.** *Let  $G$  be the graph given below.*



*We will now show that the Eliahou-Kervaire formula holds for the Betti numbers if we split along the four cycle. Let  $I = \langle u_1 u_2, u_2 u_3, u_3 u_4, u_1 u_4, u_1 w_1, u_2 w_1 \rangle$ . Then, let  $J = \langle u_1 u_2, u_2 u_3, u_3 u_4, u_1 u_4 \rangle$ , and  $K = \langle u_1 w_1, u_2 w_1 \rangle$ . It is now easy to show that  $J \cap K = \langle u_1 u_2 w_1, u_2 u_3 w_1, u_1 u_4 w_1 \rangle$ . Then we have,*

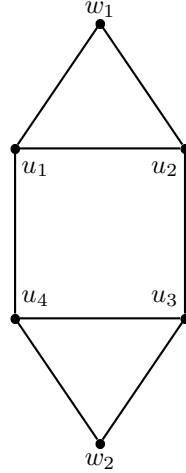
	$\beta(I)$				$\beta(J)$				$\beta(K)$			$\beta(J \cap K)$	
	1	2	3		1	2	3		1	2		1	2
1	6	8	3	1	4	4	1	1	2	1	1	-	-
total	6	8	3	total	4	4	1	total	2	1	total	3	2

One can easily see that the Betti numbers sum as indicated in Theorem 2, while  $G$  has adjacent vertices of degree three (namely,  $u_1$  and  $u_2$ ).

We should say that it is probably not a surprise that such examples exist. It has long been known that there are ideal decompositions for which one can show that no splitting function exists, but for which the formula for the Betti numbers given in Theorem 2 holds.

We now want to illustrate that the construction of a splitting that we gave above fails for graphs not satisfying the hypotheses of our proposition. In particular, we consider a graph having adjacent vertices of degree greater than two. We then demonstrate that there is no splitting function if one wishes to split along the cycle. Furthermore, we show that the Betti number formula of Eliahou and Kervaire does not hold for the edge ideal of this graph.

**Example 8.** Consider the graph,  $G$ :



Then  $V = \{u_1, u_2, u_3, u_4, w_1, w_2\} = \{u_1, u_2, u_3, u_4\} \sqcup \{w_1, w_2\} = U \sqcup W$ , and  $E = E_U \sqcup E_W \sqcup E_X$  where  $E_U = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$ ,  $E_W = \emptyset$ ,  $E_X = \{u_1w_1, u_2w_1, u_3w_2, u_4w_2\}$ .

Then the edge ideal is given by,

$$\begin{aligned} I := \mathcal{I}(G) &= \langle u_1u_2, u_2u_3, u_3u_4, u_4u_1, u_1w_1, u_2w_1, u_3w_2, u_4w_2 \rangle \\ &= \langle E_U \sqcup E_W \sqcup E_X \rangle. \end{aligned}$$

Then, if we wish to split along the 4-cycle, we must take

$$J := \langle u_1u_2, u_2u_3, u_3u_4, u_4u_1 \rangle = \langle E_U \rangle,$$

and

$$K := \langle u_1w_1, u_2w_1, u_3w_2, u_4w_2 \rangle = \langle E_X \sqcup E_W \rangle .$$

Then it is easy to see that  $\mathcal{G}(J \cap K) = \{u_1u_2w_1, u_2u_3w_1, u_2u_3w_2, u_1u_4w_1, u_1u_4w_2\}$ .

We now will show that there exists no splitting function for this decomposition of the edge ideal (i.e.,  $I = J + K$  and  $\mathcal{G}(I) = \mathcal{G}(J) \sqcup \mathcal{G}(K)$ ). We begin trying to construct a function  $\mathcal{G}(J \cap K) \rightarrow \mathcal{G}(J) \times \mathcal{G}(K)$  satisfying property (a) of Definition 1. It's quite easy to see that this requirement forces us to define

$$\begin{aligned} \mathcal{G}(J \cap K) &\rightarrow \mathcal{G}(J) \times \mathcal{G}(K) \\ u_1u_2w_1 &\mapsto \begin{pmatrix} u_1w_1 \\ u_1u_2, \text{ or} \\ u_2w_1 \end{pmatrix} \\ u_1u_4w_1 &\mapsto (u_1u_4, u_1w_1) \\ u_2u_3w_1 &\mapsto (u_2u_3, u_2w_1) \\ u_1u_4w_2 &\mapsto (u_1u_4, u_4w_2) \\ u_2u_3w_2 &\mapsto (u_2u_3, u_3w_2) \end{aligned}$$

Now, choose  $S = \mathcal{G}(J \cap K)$  and we have

$$\begin{aligned} \text{lcm}(\psi(S)) &= u_1u_2u_3u_4w_1w_2 \\ &= \text{lcm}(S). \end{aligned}$$

Note that the least common multiple that we computed is the same no matter what choice we make for the image  $\psi(u_1u_2w_1)$ . Thus, there is no splitting function if one decomposes the edge ideal along the 4-cycle.

This example also illustrates why the hypotheses of Proposition 6 are necessary. That is, it is precisely the requirement that adjacent vertices of the cycle are not both connected to the complement of the cycle that guarantees that the property (b) of Definition 1 holds.

We now also show that, unlike Example 7, the Betti numbers do not sum as in the Elicahou-Kervaire formula. Then, one computes the Betti numbers of  $I$ ,  $J$ , and  $K$  to be:

	$\beta(I)$			
	1	2	3	4
1	8	12	5	-
2	-	2	4	2
total	8	14	9	2

	$\beta(J)$		
	1	2	3
1	4	4	1
total	4	4	1

	$\beta(K)$			
	1	2	3	4
1	4	2	-	-
2	-	4	4	1
total	4	6	4	1

The Betti numbers of  $(J \cap K)$  are:

	1	2	3
1	-	-	-
2	6	6	-
3	-	-	1
total	6	6	1

Now, one can easily see that, for  $\beta_4$ , the Eliahou-Kervaire formula holds:

$$\beta_4(I) = \beta_4(J) + \beta_4(K) + \beta_3(J \cap K),$$

However, the formula does not hold otherwise:

$$\beta_3(I) \neq \beta_3(J) + \beta_3(K) + \beta_2(J \cap K)$$

$$\beta_2(I) \neq \beta_2(J) + \beta_2(K) + \beta_1(J \cap K).$$

Thus, the cycle is neither splitting nor does the sum formula for Betti numbers hold.

Thus, we have shown a sufficient condition for a cycle to be splitting, and provided an example showing that our condition is not also necessary. If one hopes to use our result (together with the result of Hà and Van Tuyl, [2]) to provide a recursive algorithm for the computation of the Betti numbers of graphs that are not chordal, but that contain a cycle that satisfies the hypotheses of Proposition 6, then one must be able to compute  $\beta_{i-1}(J \cap K)$  for these cycles. This is the problem to which we are currently turning our attention.

Though a recursive algorithm to compute the Betti numbers of graphs that are not chordal is still out of reach, as an application of our proposition we can now give the Betti numbers for a special class of graphs that are not chordal. In the following example, we derive the formula for the Betti numbers of a wheel graph with an odd number of vertices and every other spoke missing.

**Example 9.** As a non-trivial example, consider the graph defined as follows:  $G = (V, E)$  with  $V = \{w, u_1, \dots, u_{2k}\}$ ,  $E = \{u_1u_2, \dots, u_{2k-1}u_{2k}, u_{2k}u_1, wu_2, wu_4, \dots, wu_{2k}\}$ . This is a wheel graph with an odd number of vertices and every other spoke missing. In this case,  $J$  corresponds to  $C_{2k}$  and  $K$  to  $S_k = \text{star}(k)$ , the subgraph consisting of the hub and remaining  $k$  spokes of the wheel. It is easy to see that  $J \cap K = wI(C_{2k})$ . Thus  $\beta_{i,j}(G) = \beta_{i,j}(C_{2k}) + \beta_{i,j}(S_k) + \beta_{i-1,j}(C_{2k})$ . Formulas for each of these is known, see [4]. Putting this together with the recursive algorithm of [2], we see that for  $k > 1$ , the wheel graph  $W(n)$  on  $n = 2k + 1$  vertices has Betti numbers given by

$$\beta_{i,j}(W) = \begin{cases} \beta_{i,j}(C_{2k}) + \beta_{i,j}(S_k) + \beta_{i-1,j}(C_{2k}) + \sum_{a=k}^{2k-1} \binom{a}{i}, & j = i + 2 \\ \beta_{i,j}(C_{2k}) + \beta_{i,j}(S_k) + \beta_{i-1,j}(C_{2k}), & \text{else} \end{cases}$$



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